Midterm - Partial Differential Equations (2022-23)
Time: 2 hours.
Attempt all questions, giving proper explanations.
You may quote any result proved in class without proof.

1. [**3 marks**] Give an example to show that non-uniqueness might hold for differential equations of the form $u'(t) = f\big(u(t)\big)$ with continuous $f : \mathbf{R} \to \mathbf{R}$, unless some additional assumptions are made on f.

2. Consider the differential equation $u'(t) = f\big(u(t)\big)$, where $f : \mathbf{R} \to \mathbf{R}$ is Lipschitz.
   (a) [**2 marks**] Show that for any $t_0 \in \mathbf{R}$, $u(t)$ is a solution if and only if $u(t - t_0)$ is a solution.
   (b) [**5 marks**] Show that for two maximal solutions $\{x_j(t),\ t \in I_j\}$, $j = 1, 2$, the images $\gamma_j = \{x_j(t) : t \in I_j\}$ either coincide or are disjoint.

3. [**9 marks**] Find the solution of the heat equation $\partial_t u = \partial_{xx} u$ on the spatial interval $[0, \pi]$, with boundary conditions $\partial_x u(t, 0) = 0$ and $u(t, \pi) = 0$, and initial profile $u(0, x) = f(x)$ satisfying the boundary conditions. Clearly state any assumptions you might make in solving the problem.

4. [**11 marks**] Solve the PDE $u_t + uu_x = 0$, with initial profile

$$u(0,x) = \begin{cases} -1, & -\infty < x \leq -a, \\ x/a, & -a < x < a, \\ 1, & a \leq x < \infty. \end{cases}$$

   If no continuous solution exists, find a weak solution.

   Next solve the PDE $u_t + uu_x = 0$, with the (different from previous question) initial profile

$$u(0,x) = \begin{cases} 1, & -\infty < x \leq -a, \\ -x/a, & -a < x < a, \\ -1, & a \leq x < \infty. \end{cases}$$

   If no continuous solution exists, find a weak solution.